Letter to Bram Petri

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Dear Bram,

Let X denote a connected compact hyperbolic surface. Let Δ_X denote the Laplace-Beltrami operator on X. The spectrum of Δ_X is discrete and accumulates only at ∞; the bottom of the spectrum is 0 and is a simple eigenvalue due to X being connected. We denote by $\lambda_1(X) > 0$ the smallest non-zero eigenvalue of X, and $h(X)$ the Cheeger constant. Write g for the genus of X and $\mathrm{diam}(X)$ for the diameter of X. I'll write $\mathrm{inj}(X)$ for the injectivity radius of X.

In your talk at Durham in June (on the paper [2]) you mentioned briefly how to pass from having a lower bound on $\lambda_1(X)$ to having an upper bound on $\mathrm{diam}(X)$, by passing through the Cheeger constant. Before discussing this, I recall that

$$h(X) \leq 1 + o_{g\to\infty}(1) \tag{0.1}$$

and

$$\lambda_1(X) \leq \frac{1}{4} + o_{g\to\infty}(1). \tag{0.2}$$

The first inequality (0.1) is by

$$h^2(X) \leq 1 + \frac{16\pi^2}{\mathrm{diam}(X)} \tag{0.3}$$

due to Cheng [4] together with

$$\mathrm{diam}(X) \geq \log \mathrm{vol} X)(1 - o(1))$$

as you mentioned in your talk (by comparing volumes of balls in the universal cover to volumes of balls in X). The inequality (0.2) is a result of Huber [6].

So suppose we know $\lambda_1(X)$ is large, and we want to bound $\mathrm{diam}(X)$ from above. One may use the inequality of Buser [3]

$$\lambda_1(X) \leq 2h(X) + 10h^2(X)$$

to first obtain a lower bound for $h(X)$ as

$$h(X) \geq \frac{1}{20}\left(-2 + \sqrt{4 + 40\lambda_1(X)}\right)$$

(I am not sure if something better than this exists for hyperbolic surfaces). Even if $\lambda(X) = \frac{1}{4}$ is essentially optimal, this only gives

$$h(X) \geq \frac{1}{20}\left(\sqrt{14} - 2\right) \approx 0.087$$

which is far from the optimal value of $h(X)$. Continuing with this argument for general X, we can then the following result that Mirzakhani [8, §4.6] gives as an explicit version of a result of Brooks [1]: for any $r_0 > 0$,

$$\operatorname{diam}(X) \leq 2\left(r_0 + \frac{1}{h(X)}\log\left(\frac{\operatorname{vol}(X)}{2B(r_0)}\right)\right).$$

where $B(r_0)$ is the infimum of the volume of a ball of radius r_0 in X. In particular, if we choose $r_0 = \operatorname{inj}(X)$ we obtain

$$\operatorname{diam}(X) \leq 2\left(\operatorname{inj}(X) + \frac{1}{h(X)}\log\left(\frac{\operatorname{vol}(X)}{8\pi\sinh(r_0/2)^2}\right)\right)$$

and if we assume the injectivity radius is bounded from below by $c > 0$ and $h(X) \geq h > 0$ we obtain

$$\operatorname{diam}(X) \leq \frac{2}{h}\log\operatorname{vol}(X) + O_{c,h}(1). \tag{0.4}$$

Using the previous arguments, the optimal constant in from of $\log\operatorname{vol}(X)$ one can obtain, beginning with a bound on $\lambda_1(X)$, and passing through the Cheeger constant, is about 23. On the other hand, if you have a roughly optimal bound on $h(X)$, say $h(X) = 1$, then one obtains a constant 2 in front of $\log\operatorname{vol}(X)$.

My point here is that one can do better than the previous arguments starting from a lower bound on $\lambda_1(X)$. Let me explain a more flexible bound below that will also apply to finite area hyperbolic surfaces X.

At each point of x, we write $i_X(x)$ for the injectivity radius in X at x. The thick-thin decomposition is

$$X = X_{<\epsilon} \sqcup X_{\geq\epsilon}$$

where $X_{<\epsilon}$ is the collection of points in $x \in X$ for which $i_X(x) < \epsilon$.

We now make the assumption that

$$\lambda_1(X) \geq \frac{1-\delta^2}{4} \tag{0.5}$$

for $\delta \in (0,1)$. We can write $X = \Gamma\backslash\mathbb{H}$ with Γ a discrete torsion-free subgroup of $\mathrm{SL}_2(\mathbf{R})$. The geodesic flow g_t on T^1X can be identified with the action of the one-parameter group

$$a_t \stackrel{\text{def}}{=} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$. (The reason for the appearance of $t/2$ is to make the flow unit-speed).

This flow preserves a probability measure μ on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ that projects to the Riemannian probability measure ν on X and has induced uniform probability measure in the $\mathrm{SO}(2)$ fibers (here $\mathrm{SO}(2)$ acts on the right and $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}(2)$ is identified with $\Gamma\backslash X$ in the usual way). More precisely,

$$\nu = \frac{d\operatorname{vol}(X)}{\operatorname{vol}(X)}$$

where $d\operatorname{vol}(X)$ is the volume form induced by the metric of constant curvature -1 on X. Let $L^2_0(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ denote the closed subspace of $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ consisting of (classes of) functions orthogonal to constants.

Suppose we are given $\epsilon > 0$ and two points x and y in $X_{\geq\epsilon}$. The following theorem is a quantitative version by Matheus [7] of a theorem of Ratner [9] (see [5, Lemma 2.2] for this exact formulation)

Theorem 0.1. *There is a universal constant $C > 0$ such that with assumption (0.5) on $\lambda_1(X)$, for any $\mathrm{SO}(2)$-invariant vectors f_1, f_2 in $L^2_0(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$, we have*

$$|\langle a_t f_1, f_2\rangle| \leq C\|f_1\|_{L^2}\|f_2\|_{L^2} t e^{-\frac{t}{2}(1-\delta)}.$$

Let $\tilde{v_x}$ and $\tilde{v}_y$ denote the characteristic functions of balls of radius ϵ with centers x and y, respectively. The condition $x, y \in X_{\geq\epsilon}$ imply that these balls are embedded copies of balls of radius ϵ in $\mathbb{H}$ and hence have Riemannian volume

$$\kappa\sinh(\epsilon/2)^2$$

for some $\kappa = 4\pi > 0$. We lift $\tilde{v}_x$ to an $\mathrm{SO}(2)$-invariant function v_x on $\Gamma\backslash\mathrm{SL}_2(\mathbf{R})$ and similarly lift $\tilde{v}_y$ to v_y. We write

$$v_x = \left(\int v_x d\mu\right)\mathbf{1} + v_x'$$

and similarly decompose v_y. We have

$$\int v_x d\mu = \int \tilde{v}_x d\nu = \frac{\kappa\sinh(\epsilon/2)^2}{\mathrm{vol}(X)}$$

and

$$\int v_x^2 d\mu = \int \tilde{v}_x^2 d\nu = \frac{\kappa\sinh(\epsilon/2)^2}{\mathrm{vol}(X)}$$

from which it follows that

$$\|v_x'\|_{L^2} \leq \|v_x\|_{L^2} \leq \frac{\sqrt{\kappa}\sinh(\epsilon/2)}{\sqrt{\mathrm{vol}(X)}}.$$

Furthermore, by construction, $v_x' \in L^2_0(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$. The same statements all hold with x replaced by y. Since a_t preserves $L^2_0(\Gamma\backslash\mathrm{SL}_2(\mathbf{R}))$ we have

$$\begin{aligned}\langle a_t v_x, v_y\rangle &= \left(\int v_x d\mu\right)\left(\int v_y d\mu\right) + \langle a_t v_x', v_y'\rangle \\ &= \frac{\kappa^2\sinh(\epsilon/2)^4}{\mathrm{vol}(X)^2} + \langle a_t v_x', v_y'\rangle \end{aligned} \tag{0.6}$$

and by applying Theorem (0.1) we have

$$|\langle a_t v_x', v_y'\rangle| \leq C\|v_x'\|_{L^2}\|v_y'\|_{L^2} t e^{-\frac{t}{2}(1-\delta)} \leq C\frac{\kappa\sinh(\epsilon/2)^2}{\mathrm{vol}(X)} t e^{-\frac{t}{2}(1-\delta)}.$$

Combining this with (0.6) we see that

$$\langle a_t v_x, v_y\rangle > 0$$

when

$$t e^{-\frac{t}{2}(1-\delta)} < \frac{\kappa\sinh(\epsilon/2)^2}{C\mathrm{vol}(X)}.$$

Assuming $g \to \infty$ with ϵ fixed, this holds with

$$t = \frac{2}{(1-\delta)} \left(\log \left(\frac{C\mathrm{vol}(X)}{\kappa \sinh(\epsilon/2)^2} \right) + 2 \log\log \left(\frac{C\mathrm{vol}(X)}{\kappa \sinh(\epsilon/2)^2} \right) \right).$$

On the other hand, $\langle a_t v_x, v_y \rangle > 0$ entails that there is a geodesic of length t connecting the balls of radius ϵ around x and y. This implies that the diameter of the ϵ-thick part of X is

$$\mathrm{diam}(X_{\geq \epsilon}) \leq 2\epsilon + \frac{2}{(1-\delta)} \left(\log \left(\frac{C\mathrm{vol}(X)}{\kappa \sinh(\epsilon/2)^2} \right) + 2 \log\log \left(\frac{C\mathrm{vol}(X)}{\kappa \sinh(\epsilon/2)^2} \right) \right).$$

When I refer to $\mathrm{diam}(X_{\geq \epsilon})$, I a priori mean the diameter allowing paths to leave $X_{\geq \epsilon}$ in order to connect points.

To simplify the estimate above, one can fix ϵ and δ (so forcing uniform spectral gap) and let $g \to \infty$. One obtains

$$\mathrm{diam}(X_{\geq \epsilon}) \leq \frac{2}{(1-\delta)} \log \mathrm{vol}(X) + \frac{4}{(1-\delta)} \log\log \mathrm{vol}(X) + O_{\epsilon,\delta}(1).$$

This applies also to finite area hyperbolic surfaces. The optimum result one expects from this is (roughly) when $\lambda_1 = \frac{1}{4}$ and $\delta = 0$ whence one obtains

$$\mathrm{diam}(X_{\geq \epsilon}) \leq 2 \log \mathrm{vol}(X) + 4 \log\log \mathrm{vol}(X) + O_{\epsilon,\delta}(1).$$

In fact, the 4 above can be improved slightly.

Best wishes,

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